Observables in Topological Quantum Field Theory

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Summary

Knots and braids, beside being fascinating mathematical objects, are encoded in the foundations of a number of physical theories and models, either as concrete realizations of natural systems or as important conceptual tools (*cf.* for instance the atomic model based on knot theory proposed in the nineteenth century). Mathematically, knots and braids are closely interconnected: from a braid –a 'weaving' pattern of strings– we can get a multi–component knot (a 'link') simply by tying up its free endpoints, while from a knot drawn in a plane we can select portions which look like 'over' and 'under'–crossing collections of strands.
It was in the late 1980 that knot theory was recognized by Witten to have a deep, unexpected interaction with quantum field theory. In earlier periods of the history of science, geometry and physics interacted very strongly at the 'classical' level (as in Einstein's General Relativity theory), but the main feature of such a 'quantum' connection is that geometric features that do matter are global ('topological'), any other local metric details being irrelevant. To focus on this crucial point consider a knot embedded in the 3D Euclidean space: what really matters is its 'knottiness', while the length of the string and the bending or stretching of the various portions of the string itself can be changed at will (without cutting and gluing back the endpoints).
Over the years mathematicians have proposed a number of 'knot invariants' aimed to distinguish and classify systematically all possible knots. The most significant and effective invariants are polynomial expressions (in one or two variables) with coefficients in the relative integers. Since it is always possible to 'present' a knot as the closure of a braid, the *Artin braid group* on n generators (for an n–strands braid) is called into play: topological information about 'over' and 'under' crossings are translated into an algebraic setting (elementary braidings $\leftrightarrow$ generators), opening the possibility

of computing polynomial invariants of knots from 'representations' of these finitely–presented groups into suitable algebras.
It was Vaughan Jones in 1985 who discovered the most famous polynomial invariant, the Jones' polynomial [1], and Ed Witten [2] who recognized this knot invariant as naturally associated with the vacuum expectation value (v.e.v.) of a 'Wilson loop' operator defined in unitary 'topological' quantum field theories (TQFTs) of the Schwarz–type (specifically, a non–Abelian Chern–Simons theory with gauge group $SU(2)$ in the fundamental representation) in $D = 3$ spacetime dimensions. Thus, the Jones polynomial and its 'colored' extensions can be evaluated not only as 'traces' of (abstractly chosen) matrix representations of the braid group, but fully understood at a quantum field–theoretic level in the frame of Schwarz TQFTs.
* The bibliography on the subject is huge, and I do not even try to provide a complete list. A few seminal original papers, as well as the bibliographic sources of these notes, are quoted as references in the text. A brief survey on quantum invariants and more specific references are provided in the final section.
** Remarks on focused issues and applications are included and more details will be provided on request as supplementary material.

1 Topological & algebraic knot theory: basic definitions, ambient and regular isotopy, topological invariants of knots; the braid group

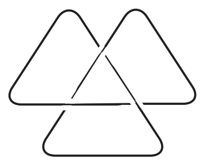
The basic references for this lecture are Joan Birman's [3, 4].

A *knot* K is defined as a continuous embedding of the circle S^1 (the 1–dimensional sphere) into the Euclidean 3–space $\mathbb{R}^3$ or, equivalently, into the 3–sphere $S^3 \doteq \mathbb{R}^3 \cup \{\infty\}$. A *link* L is the embedding of the disjoint union of M circles, $\cup_{m=1}^{M} (S^1)_m$ into $\mathbb{R}^3$ or S^3, namely a finite collection of knots referred to as the components of L and denoted by $\{L_m\}_{m=1,2,\ldots,M}$. Since each circle can be naturally endowed with an orientation, we can consider also *oriented* knots (links).

Referring for simplicity to the unoriented case, two knots K_1 and K_2 are said to be *equivalent*, $K_1 \sim K_2$, if and only if they are *ambient isotopic*. An isotopy can be thought of as a continuous deformation of the shape of, say, $K_2 \subset \mathbb{R}^3$ which makes K_2 identical to K_1 without cutting and gluing back the closed string K_2.

The *planar diagram*, or simply the *diagram*, of a knot K is the projection of K on a plane $\mathbb{R}^2 \subset \mathbb{R}^3$, in such a way that no point belongs to the projection of three segments, namely the singular points in the diagram are only transverse double points. Such a projection, together with 'over' and 'under' information at the crossing points –depicted in figures by breaks in the under–passing segments– is denoted by $D(K)$; a *link diagram* $D(L)$ is defined similarly.

In what follows we shall often identify the symbols K $[L]$ with $D(K)$ $[D(L)]$, although we can obviously associate with a same knot (link) an infinity of planar diagrams. Below: diagrams of the trefoil knot and Borromean rings (a 3-component link).



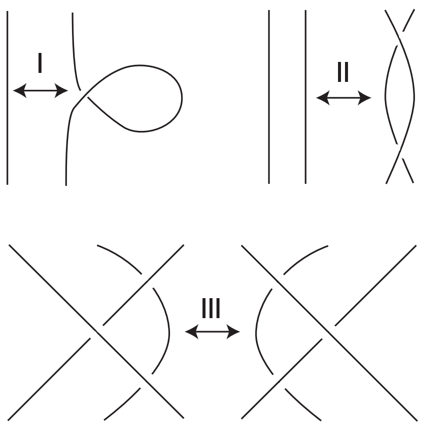
The number of crossings of a knot (diagram) is clearly a good indicator of the complexity of the knot. Indeed, Tait in late 1800 initiated a program aimed to classifying systematically knots in terms of the number of crossings (see [5] and Knot Tables on the web).

Since a knot with crossing number $\kappa \equiv c(K)$ can be presented by planar diagrams with crossing numbers $c(D(K))$ with $c(D(K)) > \kappa$ as well, the first issue to be addressed is the search for procedures aimed to simplify as much as possible the diagrams of a knot K to get a $D'(K)$ with $c(D'(K)) = \kappa$, the *minimum crossing number*. Reidemeister's theorem helps in answering to this basic question.

Equivalence of knots (Reidemeister moves). Given any pair of planar diagrams D, D' of the same knot or link, there exists a finite sequence of diagrams

$$D = D_1 \to D_2 \to \cdots \to D_k = D' \tag{1}$$

such that any D_{i+1} in the sequence is obtained from D_i by means of the three Reidemeister moves (I, II, III) depicted below.



The procedure determined by Reidemeister's theorem applies to subsets of link diagrams localized inside disks belonging to the plane where the diagram lives, and can be suitably generalized to handle diagrams of oriented links. However, notwithstanding the recursively numerable character of the implementation of the Reidmeister moves with respect to the intractability of the notion of ambient isotopy, such moves cannot be formalized into effective algorithms, basically because the above definition is purely topological. As we shall see, transformations on link diagrams can be consistently translated into an algebraic setting by exploiting their deep connection with braid groups. In the new setting the moves to be applied to link diagrams will be reformulated in terms of algebraic operations, see Markov's theorem below.

A *link invariant* is defined through a map

$$L \longrightarrow f(L), \tag{2}$$

where the quantity $f(L)$ depends only on the type of the link, namely takes different values on inequivalent links. Switching to link diagrams, we keep on using the same notation as in (2), but now it is sufficient to verify that $f(L)$ ($\equiv f(D(L))$) does not change under applications of the Reidemeister moves I, II, III.

We have already met a numerical invariant, namely the (minimum) crossing number κ. It is a natural number which takes the value 0 for the trivial knot represented as an unknotted circle.

Other invariants taking values in $\mathbb{Z}$ (relative integers) can be defined for oriented link diagrams, where each crossing is marked by ± 1 according to some fixed convention. For instance, the *writhe* $w(D(L))$ of a diagram D of an oriented link L is the summation of the signs of the crossings of D, namely

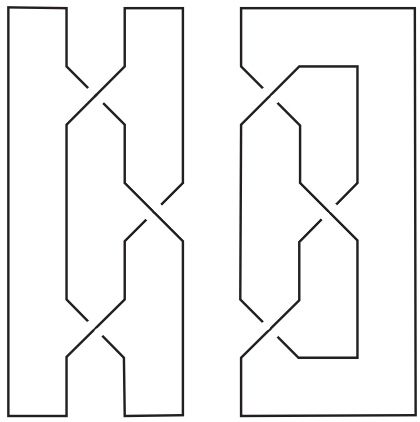
$$w(D(L)) = \sum_p \epsilon_p \equiv w(L)\,, \tag{3}$$

where the sum runs over the crossing points $\{p\}$ and $\epsilon_p = +1$ if the (directed) knot path shows an overpass at the crossing point p, $\epsilon_p = -1$ for an underpass. Note however that both the crossing number and the writhe do change under Reidmeister move of type I, but are invariant under the moves II and III: this property defines a restricted kind of isotopy, referred to as *regular isotopy*. The concept of regular isotopy is very useful because, by eliminating the move I, we do not really lose any information about the topology of the link. Moreover, the evaluation of crossing numbers and writhes can be carried out combinatorially by a simple inspection of the diagrams.

Over the years, mathematicians have provided a number of knot invariants, by resorting to topological, combinatorial and algebraic methods. Nevertheless, we do not have yet a complete invariant (neither a complete set of invariants) able to characterize the topological type of each knot and to distinguish among all possible inequivalent knots. As a matter of fact, the most effective invariants have an algebraic origin, being closely related to the braid group and its representation theory. It is straightforward to obtain a knot (link) out of a 'braid'. The inverse process is governed by Alexander's theorem.

Braids from links: Alexander's theorem. Every knot or link in $\mathbb{R}^3$ or $S^3 = \mathbb{R}^3 \cup \{\infty\}$ can be presented as a closed braid, although not in a unique way.

Actually there exist two basic types of 'closures', illustrated below for the trefoil knot: on the left the so–called platting (which requires always an even number of strands) and on the right the standard one, where the strands are tied on one side of the weaving pattern.



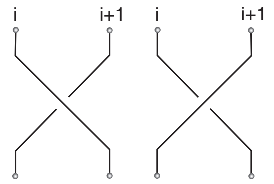
Note in passing that the trefoil (the simplest non trivial knot in Knot Tables, denoted 3_1) is an alternating knot, namely over- and under–crossing are encountered alternatively when traveling along its strand. There exists only another knot with 3 crossings, the mirror image of the trefoil, denoted 3_2: it can be checked that it is not equivalent to 3_1, and it was a byproduct of the seminal paper by Jones [1] the possibility of distinguishing them through different Jones' polynomials [see Remark 2A].

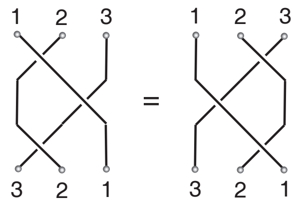
The *Artin braid group* $\mathbf{B}_n$, whose elements are (open) braids β, is a finitely presented group[1] on n standard generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$, plus the identity element e, which satisfy the defining relations

$$\begin{aligned} \sigma_i \, \sigma_j \; &= \sigma_j \, \sigma_i \quad (i, j = 1, 2, \ldots, n-1) \;\text{ if }\; |i-j| > 1 \\ \sigma_i \, \sigma_{i+1} \, \sigma_i \; &= \sigma_{i+1} \, \sigma_i \, \sigma_{i+1} \quad (\, i = 1, 2, \ldots, n-2) \end{aligned} \tag{4}$$

[1]Generators, together with algebraic relations among them, constitute a *presentation* of the given (finite or finitely–generated) group. The term 'presentation' will be used *also* in a topological context, see footnote 2 in the next lecture.

This group acts naturally on topological sets of n disjoint strands with fixed endpoints lying on two parallel lines, running downward and labeled from left to right by convention. Then each generator σ_i corresponds to the crossing of two contiguous strands labeled i and $(i+1)$, respectively; if σ_i stands for the crossing of the i–th strand over the $(i+1)$–th one, then σ_i^{-1} represents the inverse operation with $\sigma_i\,\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = e$.





An element of the braid group can be thought of as a 'word', such as for instance $\beta = \sigma_3^{-1}\sigma_2\,\sigma_3^{-1}\sigma_2\,\sigma_1^3\,\sigma_2^{-1}\sigma_1\sigma_2^{-2} \in \mathbf{B}_4$; the length $|\beta|$ of the word β is the number of its letters, where by a 'letter' we mean one of the generators or its inverse element. By a slight change of notation, denote by R_{ij} the over–crossing operation acting on two strands the endpoints of which are labeled by i and j. Then the second relation in (4) can be rewritten as

$$\mathsf{R}_{12}\,\mathsf{R}_{13}\,\mathsf{R}_{23} \;=\; \mathsf{R}_{23}\,\mathsf{R}_{13}\,\mathsf{R}_{12} \tag{5}$$

so that it corresponds to the drawing above where operations are ordered downward. Note also that this picture can be viewed as a portion of an n–strands configuration (and thus $\{1,2,3\}$ may actually represent labels attached to any triad of contiguous strands) since the first relation in (4) ensures that other types of crossings cannot happen at all.

The relation (5) is referred to as the algebraic *Yang–Baxter relation* and (in suitable forms) it characterizes the structure of solvable models in statistical mechanics and many other completely integrable (classical and quantum) systems.

As mentioned before, it is straightforward to get a link out of a braid: we have simply to close up the ends of an open braid β to get a *closed braid* $\hat{\beta}$ that reproduces the diagram of some link L. Formally

$$\beta \xrightarrow{\text{closure}} \hat{\beta} \longleftrightarrow L\,. \tag{6}$$

As shown in the picture above, such an operation can be performed in two ways, denoted $\hat{\beta}^{\,\mathrm{st}}$ (the standard closure) and $\hat{\beta}^{\,\mathrm{pl}}$ (the plat closure).

As already pointed out, Alexander's theorem does not establish a one–to–one correspondence between links and braids. For instance, given a closed braid $\hat{\beta} = L$ with $\beta \in \mathbf{B}_n$, any other braid obtained from β by *conjugation*, namely $\beta' = \alpha\beta\alpha^{-1}$ (for some $\alpha \in \mathbf{B}_n$) has a closure $\hat{\beta}'$ which reproduces the same link L. However, there exists a classical algorithm which performs the reduction in a number of steps which is bounded from above by a polynomial function of the braid index, where the *braid index* of a braid or closed braid is simply the number of its strands.

For what concerns the issue of knot equivalence, Reidemeister's theorem can be rephrased for braids according to Markov's theorem. The following statement refers to the case of open braids, which captures the crucial features of the construction, while the version involving closed braids can be found in [4], section 2.

> **Equivalence of braids (Markov moves).** Two braids are equivalent if they differ by a finite sequence of Markov moves of the following two types, together with their inverse moves:
> **i)** change a braid $\beta \in \mathbf{B}_n$ to a conjugate element in the same group, $\beta \to \alpha\beta\alpha^{-1}$, with $\alpha \in \mathbf{B}_n$;
> **ii)** change $\beta \in \mathbf{B}_n$ to $\mathsf{i}_n\,(\beta)\sigma_n^{\pm 1}$, where $\mathsf{i}_n : \mathbf{B}_n \to \mathbf{B}_{n+1}$ is the natural inclusion obtained by disregarding the $(n+1)$th strand and $\sigma_n, \sigma_n^{-1} \in \mathbf{B}_{n+1}$.

Remark 1A. We may ask whether it would be possible to select the most economical presentation of a knot diagram as a closed braid. The relevant quantity would be the *minimum braid index* of a link L, namely the minimum integer n for which there exists a braid $\beta \in \mathbf{B}_n$ whose closure $\hat{\beta}$ is a diagram for L. No explicit algorithm for this problem is known, so that its computational complexity class is presumably (more than) exponential.
Algorithmic questions concerning knot theory, knot invariants and braid group are addressed in [4]; for applications in the context of quantum computing, see a few papers by the lecturer (`arXiv`, 2006-2009) and references therein. ■

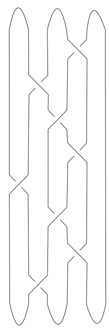
2 Topological invariants of knots from representations of the braid group in a Hecke algebra; skein relations and a glimpse of recursive combinatorial construction

The basic reference for this lecture is again [4].

Invariants of knots (links) of polynomial type arise (or can be reformulated) by resorting to *representations* of the braid group[2].

Generally speaking, in order to represent the finitely presented group $\mathbf{B}_n$ defined in (4), we need an 'algebra' structure A, namely a vector space over some field (or ring) Λ, endowed with a multiplication satisfying associative and distributive laws. The algebra must have a unit with respect to multiplication and for our purposes must be also finitely generated, namely its elements can be decomposed in terms of some finite basis set, the number of elements of which equals the braid index n.

The reason for considering an algebra should become clear if we recognize, on the one hand, that we can multiply braids $\in \mathbf{B}_n$ by simply composing their diagrams: given β_1 and $\beta_2 \in \mathbf{B}_n$ we get the product $\beta_1\, \beta_2$ by placing the braid β_1 above β_2 and gluing the bottom free ends of β_1 with the top ends of β_2 (this operation was implicitly assumed in (4) and (5), see also the central portion of the figure below).



[2] Here the term *presentation* belongs to the branch of geometry called 'geometric topo–logy', where the existence of neither a (Riemannian) metric nor a differentiable structure is required. So far we have encountered the presentation of a knot as a braid and, for 3-manifolds, there exist several types of presentations, *e.g.* by surgery along links, see Remark 5A in the final section. On the other hand, a *representation* is a homomorphism (linear map) between algebraic structures.

The operation to be associated with 'addition' of braids can be defined in terms of formal combinations of the type $a\beta_1 + b\beta_2$, for any $\beta_1, \beta_2 \in \mathbf{B}_n$ and $a, b \in \Lambda$ (the field of scalars associated with the algebra A).
With these premises, a *representation of* $\mathbf{B}_n$ *inside the algebra* A is a map

$$\rho_{\mathsf{A}} \ : \mathbf{B}_n \ \longrightarrow \ \mathsf{A} \tag{7}$$

which satisfies

$$\rho_{\mathsf{A}}(\beta_1 \beta_2) \ = \ \rho_{\mathsf{A}}(\beta_1)\rho_{\mathsf{A}}(\beta_2) \quad \forall \beta_1 , \beta_2 \in \mathbf{B}_n, \tag{8}$$

namely ρ_{A} is a group homomorphism from $\mathbf{B}_n$ to the multiplicative group $\mathsf{G} \subset \mathsf{A}$ of the invertible elements of A (in particular: $\rho_{\mathsf{A}}(e) = 1$, where e is the identity element of $\mathbf{B}_n$ and 1 denotes the unit of A; $\rho_{\mathsf{A}}(\beta^{-1}) = [\rho_{\mathsf{A}}(\beta)]^{-1}$, $\forall \beta$). By using the standard generators of $\mathbf{B}_n$ defined in (4), it suffices to define the map (7) on the generators $\{\sigma_i\}$

$$\rho_{\mathsf{A}}(\sigma_i) \ \doteq \ g_i \ \in \mathsf{G} \ \subset \ \mathsf{A}\,, \quad (i = 1, 2, \ldots n-1)\,, \tag{9}$$

and to extend linearly its action on products and sums of braids. Therefore any pair of contiguous elements g_i and g_{i+1} must satisfy the *Yang–Baxter equation associated with the representation* ρ_{A}, namely

$$g_i \, g_{i+1} \, g_i \ = \ g_{i+1} \, g_i \, g_{i+1} \tag{10}$$

while $g_i g_j \ = \ g_j g_i$ for $|i-j| > 1$. To make the algebra A a *Hecke algebra* it is necessary to add the requirement

$$\mathsf{A} \ \mapsto \ H_n\,(t) \ \text{ with } \ (g_i)^2 \ = \ (t-1)\, g_i \ + \ t\,. \tag{11}$$

Here t is an element of Λ^* and for $t = 1 \in \Lambda^*$, $H_n\,(1)$ is isomorphic to the group algebra of the symmetric group on n objects, Σ_n.
[Σ_n can be generated by $\{\tau_i\}$, $i = 1, 2, \ldots, n-1$, the swaps of pairs of contiguous objects, with relations: $(\tau_i)^2 = 1\,, \forall i$, 1 being the identity permutation. Thus we can think of $H_n\,(t)$ as encoding a 'deformation' of the group algebra of the symmetric group: the choices and the role of such deformation parameter will become clear in what follows.]

For simplicity, we keep on using A and introduce *matrix representations* of some fixed dimension N by representing A over the algebra of $(N \times N)$ matrices with entries in the field Λ

$$\mathsf{A} \ \longrightarrow \ \mathsf{M}(\Lambda\,, N). \tag{12}$$

If we restrict the domain of the above map to the group $\mathsf{G} \subset \mathsf{A}$ of invertible elements, the assignment (12) can be rephrased as the choice an N–dimensional vector space V over Λ, and thus we have the natural isomorphism

$$\mathsf{M}(\Lambda\,, N) \ \cong \ \mathsf{GL}_{\Lambda}\,(V,\, N), \tag{13}$$

where $\mathsf{GL}_\Lambda\,(V,\,N)$ is the general linear group of non–singular, Λ–linear maps $V \to V$.

Loosely speaking, if we associate with a braid $\beta \in \mathbf{B}_n$ a matrix $M(\beta)$ obtained by means of a representation (12) of dimension $N = n$, then β can be characterized by a scalar, namely the trace of $M(\beta)$ (the character of the representation in a group–theoretic language). Such traces are candidates to be interpreted as invariants of links presented as closed braids. More formally, a *trace function over the algebra* A is defined as a linear function over A and, by extension, over a matrix representation algebra (12)

$$\mathsf{A} \;\longrightarrow\; \mathsf{M}(\Lambda\,,N) \;\xrightarrow{\mathrm{Tr}}\; \Lambda \tag{14}$$

satisfying the property

$$\mathrm{Tr}\,(\,M(\beta)\,M'(\beta')\,)\;=\;\mathrm{Tr}\,(\,M'(\beta')\,M(\beta)\,). \tag{15}$$

for any $M(\beta)\,,M'(\beta')$ which are the images under ρ_{A} of two braids β,β' $\in \mathbf{B}_n$. Then $\mathrm{Tr}(M(\beta))$ is a link invariant since it does not change under Markov move of type **i)** (defined in lecture 1), namely

$$\mathrm{Tr}\,(M(\beta))\;=\;\mathrm{Tr}\,(M'(\beta'))\quad \text{if } \beta \text{ and } \beta' \text{ are conjugate.} \tag{16}$$

Link invariants arising as Markov traces of representations of $\mathbf{B}_n$ in $H_n\,(t)$ are then polynomials in the *formal variable* t and turn out to be 'regular' isotopy invariants (we leave aside the issue of the further invariance under Markov moves of type **ii)** which would provide 'ambient' isotopy invariants).

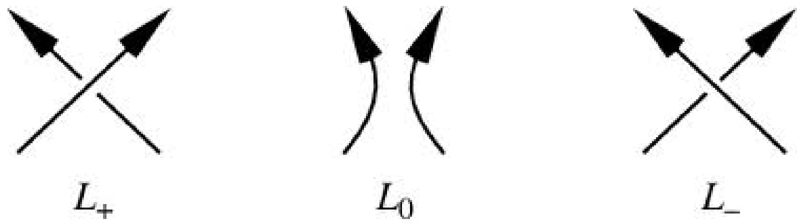
The general algebraic setting outlined above underlies the constructions of both the Jones [6] and HOMFLY [7] link polynomials. In particular:

- the *Jones polynomial* of a link L, $\mathsf{J}(L;t)$, is the Markov trace of the representation of $\mathbf{B}_n$ in the Hecke algebra $H_n(t)$ [3]. It is a Laurent polynomial in one formal variable t with coefficients in $\mathbb{Z}$, namely it takes values in the ring $\Lambda \equiv \mathbb{Z}[t,t^{-1}]$;
- the *HOMFLY polynomial* $P(L;t,z)$ is obtained as a one–parameter family of Markov traces (parametrized by a further $z \in \Lambda^*$) of the representation of $\mathbf{B}_n$ inside the Hecke algebra $H_n(t)$. It is a Laurent polynomial in two formal variables with coefficients in $\mathbb{Z}$, namely it takes values in $\Lambda \equiv \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$.

[3] The original definition in [1] was formulated in terms of the Temperley–Lieb algebra $TL_n(t)$, sometimes referred to as Jones' algebra. However, it can be shown that the irreducible representations of this algebra are in 1–1 correspondence with those of the Hecke algebra of the same dimension n, see [4] section 4.3.

Remark 2A:
skein relations and recursive combinatorial constructions

As pointed out by Jones himself [8], the simplest way of calculating polynomials of knots, at least for low crossing numbers, is by means of *linear skein relations*, which are at the basis of the combinatorial constructions introduced by Conway in the early 1960s and fully developed by Kauffman (the 'bracket' method) [9]. We give here a very brief account on this, referring to oriented diagrams. According to the standard notation, we denote by the symbol L a general diagram (of a knot or link), and by L_+, L_0, L_- respectively the local configurations which may occur within L:



The basic skein relation is given by the three–term linear recursion formula

$$a\, \mathsf{J}(L_+) \,+\, a'\, \mathsf{J}(L_-) \,=\, d\, \mathsf{J}(L_0) \tag{17}$$

where J denotes the polynomial in question and $a,\ a'$, d are non–null scalars $\in \Lambda^*$. In most cases $a' = (a)^{-1}$ and they are related to the (single) variable of the polynomial (d is a suitable combination of powers of the other parameters). The skein relation, together with the choice of a normalization for the unknot (the circle) $L_{\bigcirc}$,

$$\mathsf{J}\,(L_{\bigcirc}) \,=\, b\ \in \Lambda^* \tag{18}$$

can be shown to provide, through a recursive procedure of 'disentanglement' of the diagram, the expression of the polynomial $\mathsf{J}\,(L)$, invariant under ambient isotopy (Reidemeister moves, see lecture 1)[4].

■

[4]Note that in this combinatorial construction the skein relation plays the role of 'defi–ning relation' for the invariant, leaving somehow aside the issue of representations of the braid group in some (Hecke or other types of) algebras. However, these approaches are not really independent, and we will come back on this later on, in connection with the TQFT definition of the knot invariants in lecture 4.

Remark 2B:
the Jones polynomial for the trefoil knots

The skein relation for the Jones polynomial reads (we keep on using J although the symbol used by Jones is $V(L;t)$)

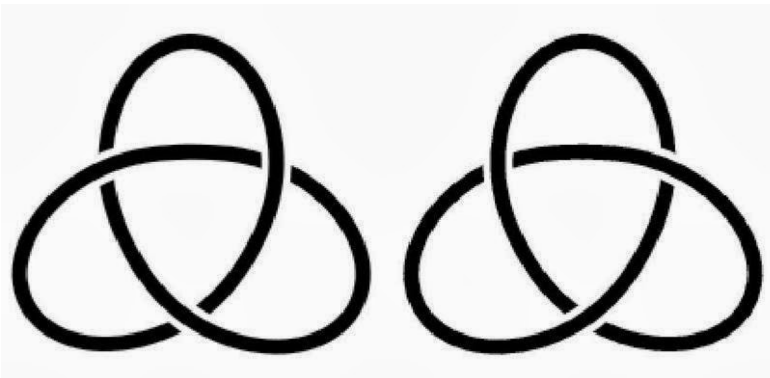
$$\frac{1}{t}\,\mathsf{J}(L_+) \,-\, t\,\mathsf{J}(L_-) \,=\, (\sqrt{t} \,+\, \tfrac{1}{\sqrt{t}})\;\mathsf{J}(L_0) \tag{19}$$

with the normalization

$$\mathsf{J}\,(L_{\bigcirc}) \,=\, 1 \tag{20}$$

so that it can be considered as a Laurent polynomial in the formal variable $\sqrt{t}$ (the ring of such polynomials can be still denoted $\Lambda \equiv \mathbb{Z}[t, t^{-1}]$ as done in the definition trough the Hecke algebra).

As an illustrative example, below we show the standard, right–handed trefoil knot 3_1 (left) and its mirror, left–handed image 3_2 (right).



The knots have to be thought of as oriented in order to apply the procedure of Remark 2A, although it can be shown that the value of the Jones' invariant is the same if we reverse the orientation of the strand of (any) knot. The trefoil is a chiral knot, as already mentioned, and the Jones' polynomial is capable of distinguishing the two configurations, unlike all previously known invariants. Indeed

$$\mathsf{J}\,(3_1\,) \,=\, t \,+\, t^3 \,-\, t^4 \tag{21}$$

$$\mathsf{J}\,(3_2\,) \,=\, t^{-1} \,+\, t^{-3} \,-\, t^{-4} \tag{22}$$

See "http://en.wikipedia.org/wiki/Skeinrelation" for explicit examples of calculations on a few knot diagrams. ■

3 Algebraic construction of colored Jones' polynomials in the $SU(2)_q$ framework; universality

The basic reference for this lecture would be the paper by Reshetikhin and Turaev [10] which, together with Witten's [2], represent the pillars of the whole subject. However, the treatment of Kirby and Melville [11] turns out to be more simple and self–contained, and will be followed in this lecture.

The general algebraic approach of lecture 2 leaves aside the crucial quest of *unitarity* for representations of the braid group $\mathbf{B}_n$ employed in evaluating link invariants. This is of course an input of physical nature, based on the fact that a number of 'quantum' (many–body) systems and field theories do support unitary representations of the braid group. The resulting 'quantum' invariants[5] have been extensively studied since the 1990s and are of continuous interest for both geometers and theoretical physicists (see more comments in the final section).

> **Universality.** These invariants arise from a number of historically distinct approaches, ranging from R–matrix representations obtained with the quantum group method (addressed here), monodromy representations of the braid group in $2D$ conformal field theories and the quasi–tensor category approach by Drinfel'd up to $3D$ quantum Chern–Simons theory (see the next lecture). All these frames share the common feature of being 'integrable' or 'solvable', a property reflected by the presence of Yang–Baxter–like equations encoding the algebraic structure of braid groups in disguise. Basically, the inherent representations are unitarily equivalent to each other.

Note preliminarly that, since we are looking for unitary representations to be associated with link invariants, the formal variable of the polynomials (including in particular the Jones polynomial) has to be a unitary complex number $c \in \mathbb{C}$, with $|c| = 1$. The commonly adopted variable is a complex, r–th *root of unity*, namely

$$q := \exp(2\pi\, i/r) \;\; , r \in \mathbb{N},\; r \geq 1 \tag{23}$$

[5] 'Quantum' has a double, interconnected meaning: it accounts for (Topological) 'Quantum' Field Theory (see lecture 4) and also for the presence of a 'quantum' group symmetry, which might be looked at as related to regularization of the quantized theory.

and the idea is that, by letting r grow, the polynomial can be evaluated in more and more points lying on the unit circle in $\mathbb{C}$. The upgraded notation for the Jones polynomial is

$$\mathsf{J}\,(L;q)\ \in\ \mathbb{Z}[q,q^{-1}]. \tag{24}$$

The invariant of an <u>oriented</u> link L we are going to address is an extension of the Jones polynomial (24), denoted by

$$\mathsf{J}\,(L;q;j_1,j_2,\ldots,j_M) \tag{25}$$

and parametrized by a set of labels $\{j_1,j_2,\ldots,j_M\}$ (the 'colors') to be assigned to each of the M link components $\{L_i\}_{i=1,2,\ldots,M}$. From the point of view of equivalence of links, $\mathsf{J}\,(L;q;j_1,j_2,\ldots,j_M)$ turns out to be a 'regular isotopy' invariant, but it can be shown that the quantity

$$\frac{q^{-3w(L)/4}}{q^{1/2}-q^{-1/2}}\ \ \mathsf{J}(L;q;j_1,j_2,\ldots,j_M), \tag{26}$$

where $w(L)$ is the writhe of the link L defined in (3) of lecture 1, is invariant under any ambient isotopy transformation.
The *colored polynomials* (26) reduce to Jones' (24) when all the colors j_1,j_2, $\ldots$, j_M are equal to a same j, with $j=1/2$, but they are genuine generalizations as far as they can distinguish knots with the same Jones polynomial.

The basic objects in the definition of the invariants (25) are oriented, colored links and braids. Recall that a link is oriented if all its components $\{L_i\}_{i=1,2,\ldots,M}$ are endowed with an orientation. Since L can be thought of as the closure $\hat{\beta}$ of an open braid $\beta\in\mathbf{B}_n$ for some n (Alexander's theorem of lecture 1), each strand of β ($\hat{\beta}$) inherits naturally an orientation, depicted in figures by an arrow. The assignment of 'colorings' can be carried out in two different ways, namely by assigning a color either to each oriented link component L_m ($m=1,2,\ldots,M$) or to each strand $\mathfrak{l}_i$ ($i=1,2,\ldots,n$) of the associated oriented braid $\beta\in\mathbf{B}_n$. Of course the braid index n is in general (much) greater than the number of link components M, as can be easily recognized from the drawing at page 9 (the Borromean link with 3 components presented as a 6-strands braid). Here we are going to illustrate the first choice[6] which is technically simpler and emphasize the role of the so–called R–matrix, namely the set of representations of the crossings in link diagrams, as 'braiding operators' over the representation ring of $SU(2)_q$ [11].

[6]Both choices of the colorings provide actually the same colored link invariants, possibly up to an overall normalization factor.

The *representation ring of* $SU(2)_q$, denoted $\mathfrak{R}\,(SU(2)_q)$, can be introduced following in the footsteps of the construction of $SU(2)$–representation theory. According to our previous notation, the ground ring (in which the link invariants will take their values) is $\Lambda = \mathbb{Z}\,[q^{\pm 1}] \subset \mathbb{C}$, with $q = \exp(2\pi i/r)$ as in (23). The elements of $\mathfrak{R}\,(SU(2)_q)$ are complex Hilbert spaces, invariant under the action of the group (recall that a vector space V is invariant under the action of a group G if $G \times V \to V$, namely transformed vectors keep on belonging to V; such spaces are referred to as invariant G–modules). As happens for $SU(2)$, it can be shown that $\mathfrak{R}\,(SU(2)_q)$, is spanned by finite–dimensional $SU(2)_q$–modules $\{V^j\}$. In the case of $SU(2)$ the labels $\{j\}$ (the spin quantum numbers from the quantum mechanical point of view) run over all integers and half–integers $\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, each V^j is characterized by its dimension $(2j+1)$ and is irreducible (namely cannot be decomposed into a direct sum of invariant subspaces of lower dimensions).

In the q–deformed case it can be shown that the $SU(2)_q$–modules $\{V^j\}$ are irreducible if and only if the labels $\{\mathbf{j}\}$ run over the finite set $\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, r\}$. Each V^j, spanned by $(2j+1)$ vectors, can be characterized by a scalar $\in \Lambda$, the q–integer $[2j+1]_q$, where $[\mathfrak{n}]_q = (q^{\mathfrak{n}/2} - q^{-\mathfrak{n}/2})/(q^{1/2} - q^{1/2})$ for $\mathfrak{n} \in \mathbb{N}^+$ a positive integer. Thus, for each choice of the integer r, we have a distingushed family of irreducible representations (irreps) of $SU(2)_q$

$$\mathfrak{F}_r \,=\, \{\, V^j \,\}_{j=0,\ldots,r} \;\; ; \;\; V^j \,\leftrightarrow\, [2j+1]_q \quad \text{(the } q\text{-dimension)} \tag{27}$$

which makes $\mathfrak{R}\,(SU(2)_q)$ a finitely generated ring. As in the case of $SU(2)$, the ring structure is made explicit in terms of the direct sum $\oplus$ and (non commutative and non associative) tensor product $\otimes$ of irreps

$$V^j \,\oplus\, V^k \,\in \mathfrak{R}\,(SU(2)_q) \text{ if } j,k \leq r$$

$$V^j \,\otimes\, V^k \,\in \mathfrak{R}\,(SU(2)_q) \text{ if } j+k \leq r, \tag{28}$$

where the ranges of the labels have to be suitably restricted with respect to the standard case. The analogue of the Clebsch–Gordan series, giving the decomposition of the tensor product of two irreps into a (truncated) direct sum of irreps, reads

$$V^{j_1} \otimes V^{j_2} \;=\; \bigoplus_{j=|j_1-j_2|}^{\min\{j_1+j_2,\, r-j_1-j_2\}} V^j. \tag{29}$$

Note however that the ring $\mathfrak{R}(SU(2)_q)$ is much richer than its ‘classical’ $SU(2)$–counterpart because $SU(2)_q$ can be endowed with a *quasi–triangular Hopf algebra* structure. This means that, besides the standard operators $\oplus$ and $\otimes$ (which obey distributive laws) we can also introduce a co–multiplication $\Delta : SU(2)_q \rightarrow SU(2)_q \otimes SU(2)_q$, an antipode map $A : SU(2)_q \rightarrow SU(2)_q$, a counit $\varepsilon : SU(2)_q \rightarrow \mathbb{C}$ and a distinguished invertible element

$$\mathsf{R} \in SU(2)_q \otimes SU(2)_q, \tag{30}$$

called the *R–matrix* [7]. The far–reaching role played by the R–matrix becomes manifest when we define its action on the tensor product of a pair of irreducible $SU(2)_q$–modules in $\mathfrak{R}(SU(2)_q)$. Denoting by $\hat{\mathsf{R}}$ the operator associated to R, we have

$$\hat{\mathsf{R}} : \; V^j \otimes V^k \longrightarrow V^k \otimes V^j, \tag{31}$$

where, according to (28) the values of the labels j, k have to be suitably restricted. These $\hat{\mathsf{R}}$–operators will be referred to as *braiding operators* associated with the R–matrix (30). If we further extend the action of $\hat{\mathsf{R}}$ to the ordered product of three irreps $V^j \otimes V^k \otimes V^l$ by defining

$$\hat{\mathsf{R}}_{jk} \doteq \hat{\mathsf{R}} \otimes \mathrm{Id} : \; (V^j \otimes V^k) \otimes V^l \longrightarrow (V^k \otimes V^j) \otimes V^l$$

$$\hat{\mathsf{R}}_{kl} \doteq \mathrm{Id} \otimes \hat{\mathsf{R}} : \; V^j \otimes (V^k \otimes V^l) \longrightarrow V^j \otimes (V^l \otimes V^k)\,, \tag{32}$$

where Id is the identity operator on the corresponding factor, then it can be shown that these operators satisfy the *quantum* Yang–Baxter equation

$$\hat{\mathsf{R}}_{jk}\, \hat{\mathsf{R}}_{kl}\, \hat{\mathsf{R}}_{jk} = \hat{\mathsf{R}}_{kl}\, \hat{\mathsf{R}}_{jk}\, \hat{\mathsf{R}}_{kl}\,. \tag{33}$$

The adjective ‘quantum’ refers here to the underlying quantum group setting, and it easily recognized that (33) maps to (5), the algebraic Yang–Baxter relation, if we perform the substitutions

$$\begin{array}{lcl} \text{ordered triple}\,(j\,k\,l) & \longmapsto & \text{ordered triple}\,(1\,2\,3) \\ \hat{\mathsf{R}}\,(\text{braiding operator}) & \longmapsto & \mathsf{R}\,(\text{crossing})\,. \end{array} \tag{34}$$

The explicit expression of the braiding operator $\hat{\mathsf{R}}$ (and of its inverse $\hat{\mathsf{R}}^{-1}$) can be worked out explicitly by selecting orthonormal basis sets in the $SU(2)_q$–modules V^j, V^k, for each admissible choice of the pair j, k. For (all choices of) such bases, the braiding operators (31) and (32) are *unitary.*

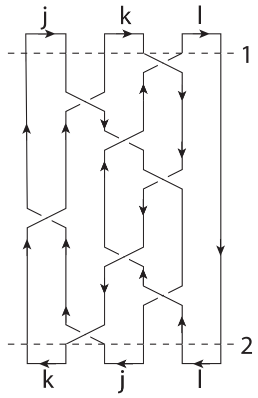
Having collected all the necessary algebraic ingredients, the colored invariant (25) for an oriented link L with M components can be now consistently interpreted in terms of a Λ–linear map

$$\mathsf{J}\,(L; q; j_1, j_2, \ldots, j_M) : \;\; \mathfrak{R}\,(SU(2)_q) \longrightarrow \Lambda\,, \tag{35}$$

[7] We do not insist any further on the explicit definitions of Δ, A and ε, refer to [11] (section 1) for more details.

where the choice of the integer r in the root of unity (23) is constrained by the requirement $r \geq M$, at least in the most general case (M distinct colors).

Remark 3A:
evaluation of $\mathsf{J}\,(L; q; j_1, \ldots, j_M)$ (more details in [12], appendix A)



In the plat presentation of the oriented, colored Borromean link as a closed braid on six strands depicted above, the parallel straight lines 1 ($\equiv \lambda_1$) and 2 ($\equiv \lambda_2$) intersect the diagram in points to be associated with Hilbert spaces which inherit the 'coloring' from the corresponding strands. The prescription for working out $\mathsf{J}\,(L; q; j_1, j_2, \ldots, j_M)$ can be summarized as follows.

- Present the link $L = \cup L_i$ ($i = 1, 2, \ldots, M$) as the plat closure of a braid and choose an orientation for each component (depicted by an arrow). Assign to each component a (distinct) color

$$L_i \quad \longrightarrow \; j_i \quad (i = 1, 2, \ldots, M)\,. \tag{36}$$

- Insert two parallel horizontal lines λ_1, λ_2 cutting the 'cap' and 'cup' portions of the diagram, respectively. This choice provide the diagram with an overall, downward orientation.
  The region of the diagram lying between λ_1 and λ_2 is an open braid whose strands inherit suitable labels from the colorings (36).

- Assign to the intersection point between a line (λ_1 or λ_2) and the string labeled by j the $SU(2)_q$ irreducible module V^j belonging to the distinguished family defined in (27).
  The whole configurations of intersection points on λ_1 and λ_2, each ordered from left to right, are to be associated with the $SU(2)_q$–modules

V_{λ_1} and V_{λ_2}, respectively, each being associated with the ordered tensor product of the individual irreps. For the 3–components Borromean link depicted above we get in particular

$$\begin{aligned} V_{\lambda_1} &= V^j \otimes V^j \otimes V^k \otimes V^k \otimes V^l \otimes V^l \\ V_{\lambda_2} &= V^k \otimes V^k \otimes V^j \otimes V^j \otimes V^l \otimes V^l \,. \end{aligned} \tag{37}$$

Note that V_{λ_1} and V_{λ_2} have the same dimension as Hilbert spaces over $\mathbb{C}$, given by the product of the dimensions of the individual factors. The number of such factors, say $2N$, is the same for the two spaces and equals the number of strands of the braid, or even the number of 'caps' ('cups') lying above the line λ_1 (below λ_2) divided by two. This feature derives of course from the topological presentation we adopted for the link L, since the braid obtained from the plat closure of any link has an even number of strands.

- Going on with the example, in the representation ring $\mathfrak{R}(SU(2)_q)$ there exists a well defined, unitary operator $\hat{\mathsf{B}}(L;q;j,k,l)$ to be associated with the transformation relating V_{λ_1} and V_{λ_2} in the diagram of the Borromean link L above

$$\hat{\mathsf{B}}(L;q;j,k,l) : V_{\lambda_1} \longrightarrow V_{\lambda_2}, \tag{38}$$

where V_{λ_1} and V_{λ_2} are explicitly defined in (37). The composite braiding operator $\hat{\mathsf{B}}(L;q;j,k,l)$ can be decomposed into an ordered sequence of the 'elementary' unitary braiding operators $\hat{\mathsf{R}}$ (and their inverses) introduced in (31), suitably tensorized with identities. The sequence is uniquely determined by going trough the diagram from λ_1 to λ_2.

- In the case of the Borromean link, the matrix elements of the braiding operator (38) evaluated on (the tensor product of) orthonormal basis vectors of the spaces V^j,V^k, V^l can be collected into a unitary $(2J+1) \times (2J+1)$ matrix parametrized by the colors j,k,l, namely

$$\mathsf{B}_{\alpha\beta}\,(j,k,l) \;\in\; U(\Lambda,\, 2J+1) \;\; (\alpha,\beta = 1,2,\ldots,2J+1), \tag{39}$$

where $U(\Lambda,\, 2J+1)$ is the algebra of unitary matrices on the ground ring $\Lambda \equiv \mathbb{Z}[q,q^{-1}]$ and $(2J+1) = (2j+1)(2k+1)(2l+1)$.
Finally, the colored link invariant $\mathsf{J}(L;q;j,k,l)$ is obtained by taking the trace of the matrix (39), formally

$$\mathsf{J}(L;q;j,k,l) \;=\; (\mathrm{Tr}\,\mathsf{B}_{\alpha\beta})\,(j,k,l), \tag{40}$$

where the resulting quantity turns out to contain the colors through the 'quantum weights' $[2j+1]_q$, $[2k+1]_q$, $[2l+1]_q$. ■

4 Holonomy in non–Abelian Chern–Simons TQFT and v.e.v. of Wilson operators; 3-manifolds and links invariants; a glimpse to other Schwarz–type TQFT

There are a number of references on this topic where either the mathematical or the field theoretic frameworks are under focus. We adopt here the latter viewpoint (closer to the original Witten's paper [2]), included in the book by Guadagnini [13] which collects also results published in papers co–authored by Martellini and Mintchev.

The axiomatic approach to TQFT due to Atiyah [14] is briefly reviewed in Remark 4A (and can be skipped by readers familiar with it).

A *topological quantum field theory* (TQFT) is a particular type of gauge theory, namely a theory quantized through the (Euclidean) path integral prescription starting from a classical Yang–Mills action defined on a suitable D–dimensional space(time). TQFT are characterized by *observables* (correlation functions, or vacuum expectation values, v.e.v.) which depend only on the global features of the space on which these theories live, namely they are independent of any metric which may be used to define the action of the classical theory. The geometric generating functionals and correlation functions of such theories are computable by standard techniques in quantum field theory and provide novel representations of certain global invariants for both D-manifolds and particular submanifolds embedded in the ambient space, such as collections of knotted curves (links). Let us recall in brief the basic axioms for a unitary TQFT in $D = 3$ before going through the case which is of interest here, namely $SU(2)$ Chern–Simons theory.

Remark 4A: axiomatic TQFT

Denote by Σ_1 and Σ_2 a pair of 2–dimensional manifolds and by $\mathcal{M}^3$ a 3–dimensional manifold with boundary $\partial\mathcal{M}^3 = \Sigma_1 \cup \Sigma_2$ (all manifolds here are compact, smooth and oriented). A unitary 3–dimensional quantum field theory corresponds to the assignment of

i) finite dimensional Hilbert spaces (endowed with non–degenerate bilinear forms) $\mathcal{H}_{\Sigma_1}$ and $\mathcal{H}_{\Sigma_2}$ to Σ_1 and Σ_2, respectively;

ii) a map (technically, a functor) connecting such Hilbert spaces

$$\mathcal{H}_{\Sigma_1} \xrightarrow{\mathbf{Z}[\mathcal{M}^3]} \mathcal{H}_{\Sigma_2} \tag{41}$$

where $\mathcal{M}^3$ is a manifold which interpolates between Σ_1 (incoming boundary)

and Σ_2 (outgoing boundary).
Without entering into details concerning a few more axioms (diffeomorphism invariance, factorization *etc.*) we just recall that unitarity implies that
iii) if $\bar{\Sigma}$ denotes the surface Σ with the opposite orientation, then $\mathcal{H}_{\bar{\Sigma}} = \mathcal{H}^*_{\Sigma}$, where $*$ stands for complex conjugation;
iv) the mappings (41) are unitary and $\mathbf{Z}[\bar{\mathcal{M}}^3] = \mathbf{Z}^*[\mathcal{M}^3]$, where $\bar{\mathcal{M}}^3$ denote the manifold with the opposite orientation with respect to $\mathcal{M}^3$.

■

The classical $SU(2)$ Chern–Simons action for the sphere S^3 (which is the simplest compact, oriented 3–manifold without boundary) is given by

$$S(A) \;=\; \frac{1}{4\pi}\int_{S^3} \mathrm{tr}\, \left(A \wedge dA \,+\, \frac{2}{3} A \wedge A \wedge A\right) \tag{42}$$

where A is the connection 1–form with value in the Lie algebra $su(2)$ of the gauge group, d is the exterior differential, $\wedge$ is the wedge product of differential forms and the trace is taken over Lie algebra indices. The partition function of the quantum theory corresponds to the map (41) restricted to the case of empty boundaries and is obtained as a 'path integral', namely by integrating the exponential of i times the classical action (42) over the space of gauge–invariant flat $SU(2)$ connections (the field variables) according to the formal expression

$$\mathbf{Z}_{CS}\,[S^3;k] \;=\; \int [DA]\, \exp\left\{\frac{i\,k}{4\pi}\, S_{CS}\,(A)\right\} \tag{43}$$

where the coupling constant k is constrained to be a positive integer by the gauge–invariant quantization procedure and is related to the deformation parameter q (see below). The generating functional (43), written for a generic closed oriented 3–manifold $\mathcal{M}^3$ is, <u>for each fixed value</u> of k, a global invariant, namely depends only on the overall topological type [8] .

Observables in the quantum theory are requested to be both gauge–invariant and invariant under diffeomorphisms (here ambient isotopies). They are expressed as (vacuum) expectation values of Wilson line operators associated with oriented knots (links) embedded in the 3–manifold, commonly

[8] The extension of (43) to the case of a manifold with boundaries, $\partial\mathcal{M}^3 \neq \emptyset$, requires modifications of the classical action (42) by suitable Wess–Zumino–Witten type terms to be associated with each boundary component. However, we do not need here the explicit expression of such boundary action since what we are interested in are expectation values of observables in the quantized field theory which require just the knowledge of (basis vectors belonging to) the boundary Hilbert spaces, *cfr.* **i)** above and Remark 4B below.

referred to as *Wilson loop operators.* Knots and link are 'colored' with irreps of the gauge group $SU(2)$, restricted to values ranging over $\{0, 1/2, 1, 3/2, \ldots, k-2\}$, where the integer $k \geq 3$ is related here to the deformation parameter by

$$q = \exp(2i\pi/k)\,. \tag{44}$$

The Wilson loop operator associated with a knot K carrying a spin–j irrep is defined as (the trace of) the *holonomy of the connection 1–form A* evaluated along the closed loop $K \subset S^3$, namely

$$\mathbf{W}_j\,[K] = \mathrm{tr}_j\, P \exp \oint_K A, \tag{45}$$

where P is the path ordering.

For a link L made of a collection of M knots $\{K_m\}$, each labeled by an irrep, the expression of the composite Wilson operator reads

$$\mathbf{W}_{j_1 j_2 \ldots j_M}\,[L] = \prod_{m=1}^{M} \mathbf{W}_{j_m}\,[K_m]. \tag{46}$$

In the framework of the path integral quantization procedure, vacuum expectation values of (all) observables are defined as functional averaging weighed with the exponential of the classical action. In particular, the v.e.v. of the Wilson operator (46) is

$$\mathcal{E}_{j_1 \ldots j_M}\,[L] = \frac{\int [DA]\, \mathbf{W}_{j_1 \ldots j_M}\,[L]\, e^{\frac{i\,k}{4\pi}\,S_{CS}\,(A)}}{\int [DA]\, e^{\frac{ik}{4\pi} S_{CS}\,(A)}}, \tag{47}$$

where $S_{CS}\,(A)$ is the CS action for the 3–sphere given in (42) and the generating functional in the denominator will be <u>normalized to 1</u> in what follows.

These Laurent polynomials (in the variable q) are actually invariants of 'regular' isotopy. Their connection with the colored Jones polynomials introduced in (26) of lecture 3 is

$$\mathsf{J}\,(L;\, q; j_1, j_2, \ldots, j_M) = \mathcal{E}_{j_1 \ldots j_M}\,[L]\,, \tag{48}$$

once suitable <u>normalizations for the unknots</u> have been chosen.
Note finally that $\mathcal{E}[L] = \mathcal{E}[\bar{L}]$, where $\bar{L}$ is obtained from L by reversing its orientation.

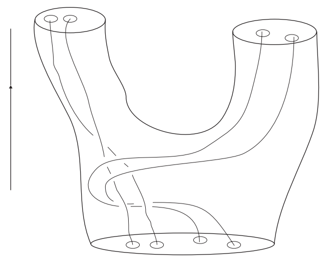
NB: again on universality of link invariants

Once given for granted the equivalence (48) (Witten's result in [2], proved in the quantum group approach by Reshetikhin and Turaev [10]) we may wonder how the explicit evaluation of (47) goes on in practice (see 4A below).

As a general remark, it can be pointed out that an *ab initio*, field theoretic strategy would consist in proving that Wilson operators such as (45) and (46) do comply with the skein relations typical of the required type of link invariants, *cfr.* Remark 2A for the Jones polynomial. The detailed proof can be found in the papers by Guadagnini, Martellini and Mintchev, published in the early 1990s and reviewed at length in the book [13].
The message we get from these results, together with those found in the 1995 paper by Cattaneo *et al.* [15] (see also [16]) are summarized as follows.
a) All polynomial invariants of oriented links (Alexander, colored Jones, HOMFLY) can be derived as v.e.v. of Wilson loops within a suitable Schwarz–type TQFT environment. Note that orientation is a quite natural requirement since we can imagine braid strands as trajectories of (quasi)particles once a suitable (2+1)D decomposition of the ambient 3-manifold is performed. The invariance is under regular isotopy (a fact which rules out Reidemeister move I), consistently with a field theoretic frame where self–intersections of trajectories are not allowed (see footnote 11).
b) Schwarz–type TQFT in D=3 are Chern–Simons (CS), with or without a cosmological constant term, and BF (a double CS, see *e.g.* [15]). In particular, colored Jones polynomials are the natural quantum observables in both CS and BF. ▲

Remark 4B: colored Jones polynomials in the quantum CS frame

The evaluation of (47) relies on the extension of CS quantum theory – endowed with a Wess–Zumino–Witten conformal field theory on its boundary– to the case in which the boundary components are intersected by knots or links, namely become 2–manifolds with punctures. The basic geometric ingredients can be easily visualized in the picture below, where a portion of a 3D manifold $\mathcal{M}^3$ (technically, a handlebody decomposition) is shown, together with an incoming boundary Σ_1 and an outgoing boundary Σ_2 made of two disjoint components, $\Sigma_2^{'}$ and $\Sigma_2^{''}$. A portion of some knot (link) embedded in the ambient 3–manifold is also depicted, and its intersections with the boundaries are 'punctures' which inherit the irreps labels from the associated (Wilson) lines. According to the axioms of TQFT, we may associate with each boundary a (finite–dimensional) Hilbert space, that is $\mathcal{H}_{\Sigma_1}$ for the incoming boundary and $\mathcal{H}_{\Sigma_2} \doteq \mathcal{H}_{\Sigma_2^{'}} \otimes \mathcal{H}_{\Sigma_2^{''}}$ (here, for simplicity, we do not explicitate the labels of punctures). The Chern–Simons(–Witten) unitary



functor (see axiom **ii)**) is then a state in the tensor product of these Hilbert spaces or, more precisely,

$$\mathbf{Z}_{CS}\,[\mathcal{M}^3\,;k]\ :\ \mathcal{H}_{\Sigma_1}\ \rightarrow \mathcal{H}_{\Sigma_2}$$

$$\Rightarrow\ \ \mathbf{Z}_{CS}\,[\mathcal{M}^3\,;k]\ \ \in\ \mathcal{H}_{\Sigma_1}\,\otimes \mathcal{H}^{*}_{\Sigma_2}, \qquad (49)$$

where in the last row we have used also axiom **iii)** since the incoming and outgoing boundaries must be endowed with opposite orientations. Moreover, such type of expression is compatible with the quantum group approach of lecture 3 because the Chern–Simons mapping in (49), once restricted to punctures, induces automatically unitary representations of the braid group in the tensor algebra of $SU(2)_q$ at a root of unity. Finally, it can be shown [14] that the conformal blocks of the $SU(2)_\ell$ Wess–Zumino field theory living on the boundaries with punctures actually provide the basis vectors for the Hilbert spaces introduced above (the level ℓ of the WZ model is related to the deformation parameter q by $q = \exp\{2\pi i/(\ell+2)\}$, and then ℓ is related to the coupling constant $k(\geq 3)$ of the CS theory in the bulk by $\ell = k-2$).

The explicit evaluation of (47) can be done by resorting, *e.g.*, to the setting given by Kaul in [16] which relies on the plat presentation of a braid whose closure is the given link L, as discussed in lecture 1. Details are omitted here and the reader may refer to [12], section 6 and appendix, for a self–contained account. ■

5 Quantum Invariants: essential bibliography and a brief review

The term 'quantum topology' was introduced by Turaev in the early 1990s to denote implications on the topological side of the algebraic theory of *quantum groups* –technically, deformations of the universal enveloping algebras of Lie groups. The latter, based on the pioneering work by Drinfel'd and Jimbo

• Drinfel'd 1987 *Quantum Groups* (Berlin–New York: W de Gruyter)
• Jimbo M 1985 A q-difference analogue of $U(g)$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** 63,

was inspired by theoretical physics from its very beginning since quantum groups and associated R–matrix representations were the basic tools of quantum inverse scattering methods and the backbone of exactly solvable models in statistical mechanics, as was soon recognized

• Wu F Y 1992 Knot theory and statistical mechanics, *Rev. Mod. Phys.* **60** 1099.
• See Martin Loebl's papers and book on this topic.

The first paper in which quantum invariants of links were addressed in the context of quantum groups is

• Kirillov A N and Reshetikhin N Y 1989 Representation of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, in *Infinite dimensional Lie algebras and groups*, *Adv. Ser. Math. Phys.* **17** 285.

The Reshetikhin and Turaev paper [10], on which lecture 3 is based, was published two years later. Quite readable accounts on the subject are

• Lickorish W 1997 *An Introduction to Knot Theory* (New York: Springer–Verlag)
• Prasolov V V and Sossinsky A B 1997 *Knots, links, braids and 3-manifolds* (Providence, RI: American Mathematical Society)

while an exhaustive collection of results, references and open problems is

• Ohtsuki T ed 2004 *Problems on invariants of knots and 3–manifolds*, RIMS Geometry and Topology Monographs **Vol. 4**, `arXiv:math.GT/0406190`.

The deformation parameter q was originally assumed to be a real number related to Planck constant by $q = e^h$, therefore it is commonly referred to as a 'quantum' deformation, while the 'classical', undeformed Lie group symmetry is recovered at the particular value $q = 1$ ($h \to 0$). As discussed in lectures 3 and 4, in dealing with quantum invariants of knots (and 3–manifolds, see below) q is most often a complex root of unity, the case $q = 1$ being considered as the 'trivial' one. However, in a topologist's language, 'classical' topological invariants are not the $(q = 1)$–counterparts of 'quantum' invariants, but rather the usual invariants of algebraic and geometric topology, typically related to the fundamental group and homology groups of manifolds and submanifolds. The standard topological invariants were *created* in order to distinguish between things and, thanks to their intrinsic definitions, it is clear what kind of properties they reflect [9]. On the other hand, quantum invariants of knots and 3–manifolds were *discovered*, but their indirect construction based on quantum group technology often hides information about the purely topological properties they are able to detect. Indeed, recognizing quantum invariants as partition functions and vacuum expectation values of physical observables in Chern–Simons–Witten (CSW) TQFT provides a 'physical' explanation of their existence and properties. Even more radically, one could speak of a 'conceptual' explanation, as far as the topological origin of these quantum invariants keeps on being unknown[10].

The CSW environment provides not only the physical interpretation of quantum invariants but it can be looked at as the prototype of all historically distinct definitions (see the remarks on *universality* in lectures 3 and 4). In particular, *monodromy* representations of the braid group appear in a variety of conformal field theories since point–like 'particles' confined in 2–dimensional regions evolve along braided world–lines.

- Kohno T 1987 Monodromy representations of braid groups and Yang–Baxter equation *Ann. Inst. Fourier* **37** 139
- Gomez C, Ruiz–Altaba M and Sierra G 1996 *Quantum group in two–dimensional physics* (Cambridge University Press) (and references therein)

As a matter of fact, the natural extension of CSW theory to a (globally hyperbolic) 3–manifold $\mathcal{M}^3$ endowed with a non empty 2D boundary $\partial\mathcal{M}^3$ induces on $\partial\mathcal{M}^3$ a specific quantized boundary conformal field theory, namely the $SU(2)$ Wess–Zumino–Witten theory at level $k + 2$, as already mentioned in lecture 4. Such extensions are strictly related to low–dimensional *gravity models* (not addressed in these lectures), see

- Witten E 1988/89 (2+1)-dimensional gravity as an exactly soluble system, *Nucl. Phys.* **B 311** 46

[9] For instance, the Euler number χ of a smooth, closed and oriented surface $\mathcal{S}$ determines completely its topological type (recall that it is defined as $\chi(\mathcal{S}) = 2 - 2g$, g being the number of 'handles' of $\mathcal{S}$).

[10] At this point we should have realized that the adjective 'quantum' includes both q-deformation of the Lie group symmetry and quantum field–theoretic frame.

• Carlip S 1998 *Quantum Gravity in 2+1 dimensions* (Cambridge University Press) (and references therein).

Remark 5A: quantum invariants of 3-manifolds presented by surgery along a link

The quantum invariants of 3–manifolds we are going to discuss in brief can be obtained as linear combinations of (colored) polynomial invariants of 'framed' unoriented links in the 3–sphere S^3 on the basis of the fact that *every closed, connected and orientable* 3*–manifold* $\mathcal{M}^3$ *can be obtained by surgery on an unoriented framed link in the* 3*–sphere*, (*cf.* the book by Lickorish (who proved this theorem), the book by Prasolov and Sossinsky and [11]). Loosely speaking, a framed oriented link $[L; \mathbf{f}]$ is obtained from L by thickening its strings to get oriented 'ribbons'. If L has M knot components $K_1, K_2, \ldots, K_M$, for each K_i we introduce another closed path K_i^f oriented in the same way as K_i and lying within an infinitesimal neighborhood of K_i[11]. Such a *presentation* of the 3-manifold, denoted $\mathcal{M}^3_L$ or $S^3 \setminus L$ in the following, is based on 'surgery' operations performed on the solid torus that can be associated with the link (the equivalence relations among surgery instructions yielding a same 3–manifold are topological operations on framed link diagrams known as *Kirby moves*). In case of an $SU(2)_q$–colored surgery link (operating in practice with a suitably chosen presentation of the link as a colored closed braid, as well as with a specific representation of braiding operators) it can be shown that the quantity

$$\mathcal{I}\,[\mathcal{M}^3_L\,;\,\mathbf{f};\,q] \;=\; \alpha^{-\sigma[L;\mathbf{f}]} \sum_{\{\mathbf{j}\}} \mu_{j_1}\,\mu_{j_2}\cdots\mu_{j_S}\; J\,[L;\,\mathbf{f},\mathbf{j}\,;\,q] \tag{50}$$

is, for each q, a topological invariant of the 3–manifold $\mathcal{M}^3_L$endowed with the framing assignment $\mathbf{f}$ (the undefined quantities are combinatorial 'weights' related to q-integers). Each $J\,[L;\,\mathbf{f},\mathbf{j}\,;\,q]$ is the 'unoriented' counterpart of the polynomial for the link L with coloring assignment $\mathbf{j}$ on its components and the summation is performed over all admissible colorings [12].
Thus colored Jones polynomials can be thought of, on the one side, as associated to CSW quantum observables given by knotted oriented trajectories on a given background manifold $\mathcal{M}^3$, and as basic bricks for all quantum invariants, on the other.

[11] It is worth noting that in the CSW environment it would have been necessary to introduce framings on knots and links, a requirement physically motivated by the general–covariant quantization procedure (see *e.g.* [13]).

[12] Unlike what happens here, where unknots should be suitably normalized, the choice of normalization for Wilson loop operators in (47) of lecture 4 was $\mathbf{Z}_{CS}\,[S^3; k] = 1$.

A detailed self–contained account on 3-manifold invariants with notations coherent with the rest of these lectures can be found in

- Garnerone S, Marzuoli A and Rasetti M 2009 Efficient quantum processing of three–manifold topological invariants, *Adv. Theor. Math. Phys.* **13** 1 (`arXiv:quant-ph/0703037`). ■

Remark 5B: perturbative invariants and the volume conjecture

In the CSW setting, 'perturbative' invariants emerge as coefficients of the asymptotic expansion of the partition function $Z_{CS}(\mathcal{M}^3_L; k)$ as $k \to \infty$ (see the review edited by Ohtsuki), or, in more physical terms, when the natural cut–off of the underlying QFT is removed. Besides a lowest order contribution that corresponds to the semiclassical (saddle point) approximation of $Z(\mathcal{M}^3_L; k)$, each other contributions in the expansion is $\exp\{2\pi\, i S_{CS}(\hat{A})/k\}$ times a power series in $1/k$ ($S_{CS}(\hat{A})$ is the CS action evaluated for the flat connection $\hat{A}$). Perturbative invariants are the coefficients of the powers $(1/k)^n$ evaluated by using $(n+1)$–loop Feynman diagrams. Although CSW is a perturbatively renormalizable quantum theory [13], the meaning of such invariants in geometric topology is only conjectured in a few cases.

In this respect, the so–called 'volume conjecture' concerns special classes of hyperbolic 3–manifolds obtained by 'surgery' along framed links in the 3–sphere and endowed with hyperbolic metrics. Focusing in particular on 'hyperbolic knots', namely those knots which give rise to <u>finite–volume</u> hyperbolic 3–manifolds, the volume conjecture stated in

- Murakami H and Murakami J 2001 The colored Jones polynomials and the simplicial volume of a knot, *Acta Math.* **186** 85

can be cast in the form

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K)|}{N} \;=\; \mathrm{Vol}\,(S^3 \setminus K)\,, \tag{51}$$

where K is the surgery hyperbolic knot and the notation $J_N(K)$ stands for the N–colored polynomial of K evaluated at $q = \exp(2\pi i/N)$.

■

The continuing interest about quantum invariants of links and 3-manifolds in field–theoretic settings even more general than 3D quantum CSW is witnessed, *e.g.*, by the recent paper

- Witten E 2014 Two lectures on the Jones polynomial and Khovarov homology, `arXiv:1401.6996[math.GT]`.

References

[1] Jones V F R 1985 A polynomial invariant for knots via von Neumann algebras, *Bull. Amer. Math. Soc.* **12** 103

[2] Witten E 1989 Quantum field theory and the Jones polynomial, *Commun. Math. Phys.* **121** 351

[3] Birman J S 1974 *Braids, links, and mapping class groups*, (Princeton University Press)

[4] Birman J S and Brendle T E 2004 Braids: a survey, in *Handbook of Knot Theory*, eds Menasco W and Thistlethwaite M (Amsterdam: Elsevier) `arXiv:mathGT/0409205`

[5] Hoste J, Thistlethwaite M and Weeks J 1998 The first 1,701,935 knots, *Math. Intell.* **20** 33

[6] Jones V F R 1987 Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** 335

[7] Freyd P, Yetter D, Hoste J, Lickorish W, Millett K and Ocneanu A 1985 A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.* **12** 183

[8] Jones V F R 2005 *The Jones polynomial*, `http://math.berkeley.edu/vfr/`

[9] Kauffman L 2001 *Knots and Physics* (Singapore: World Scientific)

[10] Reshetikhin N Y and Turaev V G 1991 Invariants of 3-manifolds via link polynomial and quantum groups, *Invent. Math.* **103** 547

[11] Kirby R and Melvin R 1991 The 3-manifold invariant of Witten and Reshetikhin–Turaev for SL(2,C), *Invent. Math.* **105** 473

[12] Garnerone S, Marzuoli A and Rasetti M 2007 Quantum automata, braid group and link polynomials *Quant. Inf. Comp.* **7** 479 `arXiv:quant-ph/0611169`

[13] Guadagnini E 1993 *The link invariants of the Chern–Simons field theory* (Berlin–New York: W de Gruyter)

[14] Atiyah M F 1989 Topological quantum field theories, *Publ. Math. IHES* **68** 175

[15] Cattaneo A S, Cotta–Ramusino P, Fröhlich J and Martellini M 1995 Topological BF theories in 3 and four dimensions *J. Math. Phys.* **36** 6137

[16] Kaul R K 1994 Chern–Simons theory, coloured oriented braids and link invariants, *Commun. Math. Phys.* **162** 289